Another Majorana Idea: Real and Imaginary in the Weinberg Theory*

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The Majorana discernment of neutrality is applied to the solutions of $j = 1$ Weinberg equations in the $(j,0) \oplus (0,j)$ representation of the Poincarè group.

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It is well known that the Dirac equation can be separated in a relativistic invariant way into a real part and an imaginary part [1]. In the hamiltonian form the real Dirac equation reads:

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \mathrm{grad}) + \beta' \mu\right] \mathcal{U} = 0 \quad , \tag{1}$$

where the Dirac matrices are chosen to be:

$$\alpha_x = \rho_1 \sigma_x; \quad \alpha_y = \rho_3; \quad \alpha_z = \rho_1 \sigma_z; \quad \beta' = -i\beta = i\rho_1 \sigma_y \quad , \tag{2}$$

and the mass term is $\mu = mc/\hbar$. The anticommutaion rules are in the configurational space

$$\mathcal{U}_i(q)\,\mathcal{U}_k(q') + \mathcal{U}_k(q')\,\mathcal{U}_i(q) = \frac{1}{2}\delta_{ik}\delta(q - q') \qquad . \quad (Eq.12\ ^{[1]}) \tag{3}$$

The hamiltonian operator is then defined

$$H = \int \mathcal{U}^\dagger \left[-c(\alpha, p) - \beta mc^2\right] \mathcal{U} dq \qquad . \quad (Eq.13\ ^{[1]}) \tag{4}$$

According to Majorana "...in the present state of our knowledge equations (12) and (13) constitute the simplest theoretical representation of a system of neutral particles". Recent indications at the mass term of neutrino [2] and at the neutrino oscillations [3] induce one to look for a formalism which could account a possible mass term and in the massless limit could lead to the Weyl scheme [4–6] in order to reproduce some predictions of the standard model.

On the other hand, recent analysis of experimental data [7] in the decays of π^- and K^+ mesons indicates the necessity of introducing tensor interactions in theoretical models. This induces one to correct our understanding the nature of the minimal coupling and to pay attention to another types of Lorentz-invariant interaction structures between the spinor and higher spin fields. In the sixties the formalism was proposed [8] for arbitrary spin-j particles, which is on an equal footing with the Dirac formalism in the $(1/2,0)\oplus(0,1/2)$ representation. It allows for other forms of Lorentz-invariant couplings [8a,§7] and [9]. Moreover, it is on an equal footing with the Dirac construct in the $(1/2,0) \oplus (0,1/2)$ representation and has no the problem of indefinite metric. The interest in this description has been considerably increased as a consequence of constructing an explicit example of the theory of the Wigner-type [10] and with the necessity of the detailed interpretation of the $E = 0$ solution of the Maxwell's equations [11–14].

In ref. [6a] the author proved that one cannot built self/anti-self charge conjugate "spinors" in the $(1,0) \oplus (0,1)$ representation. The equation $S^c_{[1]}\psi = e^{i\alpha}\psi$ has no solutions in the field of complex numbers. The $\Gamma^5 S^c_{[1]}$ self/anti-self conjugate "spinors" have been introduced there [6]. So, we have to look at alternative ways for describing $j = 1$ neutral quantum fields. The aim of this note is to apply the abovementioned Majorana idea of neutrality to the $j = 1$ states of the $(1,0) \oplus (0,1)$ representation.

In the generalized canonical (standard) representation the Barut-Muzinich-Williams matrices are expressed:

$$\overset{CR}{\gamma}_{00} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \quad \overset{CR}{\gamma}_{i0} = \overset{CR}{\gamma}_{0i} = \begin{pmatrix} 0 & -J_i \\ J_i & 0 \end{pmatrix} \quad , \tag{5a}$$

$$\overset{CR}{\gamma}_{ij} = \overset{CR}{\gamma}_{ji} = \begin{pmatrix} \eta_{ij} + \{J_i, J_j\} & 0 \\ 0 & -\eta_{ij} - \{J_i, J_j\} \end{pmatrix} \quad . \tag{5b}$$

Here $J_i, \;\; i, j = 1, 2, 3$ are the $j = 1$ matrices and $\eta_{\mu\nu}$ is the flat space-time metric. We work in the isotropic basis in which the spin matrices read

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad J_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{6}$$

By using the Wigner time-reversal operator $(\Theta_{[j]} \mathbf{J} \Theta_{[j]}^{-1} = -\mathbf{J}^*)$

$$\Theta_{[j=1]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{7}$$

one can apply the Majorana procedure to transfer over the representation where all $\gamma_{\mu\nu}$ matrices are the *real* matrices. The unitary matrix $(U^\dagger U = U U^\dagger = \mathbb{1})$ for this procedure is

$$U = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1-i) + (1+i)\Theta & -(1-i) + (1+i)\Theta \\ (1+i) + (1-i)\Theta & -(1+i) + (1-i)\Theta \end{pmatrix} \quad , \tag{8a}$$

$$U^\dagger = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+i) + (1-i)\Theta & (1-i) + (1+i)\Theta \\ -(1+i) + (1-i)\Theta & -(1-i) + (1+i)\Theta \end{pmatrix} \quad . \tag{8b}$$

As a result we arrive, $\overset{MR}{\gamma}_{\mu\nu} = U \overset{CR}{\gamma}_{\mu\nu} U^\dagger$:

$$\overset{MR}{\gamma}_{00} = \begin{pmatrix} 0 & \Theta \\ \Theta & 0 \end{pmatrix} \quad , \quad \overset{MR}{\gamma}_{01} = \overset{MR}{\gamma}_{10} = \begin{pmatrix} 0 & -J_1\Theta \\ -J_1\Theta & 0 \end{pmatrix} \quad , \tag{9a}$$

$$\overset{MR}{\gamma}_{02} = \overset{MR}{\gamma}_{20} = \begin{pmatrix} iJ_2\Theta & 0 \\ 0 & -iJ_2\Theta \end{pmatrix} \quad , \quad \overset{MR}{\gamma}_{03} = \overset{MR}{\gamma}_{30} = \begin{pmatrix} 0 & -J_3\Theta \\ -J_3\Theta & 0 \end{pmatrix} \quad , \tag{9b}$$

$$\overset{MR}{\gamma}_{ij} = \overset{MR}{\gamma}_{ji} = \frac{1}{2} \begin{pmatrix} i(J_{ij}^* - J_{ij})\Theta & (J_{ij}^* + J_{ij})\Theta \\ (J_{ij}^* + J_{ij})\Theta & -i(J_{ij}^* - J_{ij})\Theta \end{pmatrix} \quad \text{and} \quad \overset{MR}{\gamma}_5 = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix} \quad . \tag{9c}$$

Here we introduced the notation $J_{ij} = \eta_{ij} + \{J_i, J_j\}$. Since J_2 is the only (of the J_i) matrix which is imaginary in the isotropic basis we can conclude that a set of real Barut-Muzinich-Williams matrices is constructed. The $(1, 0) \oplus (0, 1)$ functions in this representation are defined

$$\overset{MR}{u}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} \phi_{_L} + \Theta\phi_{_R} \\ \phi_{_L} + \Theta\phi_{_R} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -\phi_{_L} + \Theta\phi_{_R} \\ \phi_{_L} - \Theta\phi_{_R} \end{pmatrix} = \mathcal{U}^+ + i\mathcal{V}^+ \quad , \tag{10a}$$

$$\overset{MR}{v}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} -\phi_{_L} + \Theta\phi_{_R} \\ -\phi_{_L} + \Theta\phi_{_R} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \phi_{_L} + \Theta\phi_{_R} \\ -\phi_{_L} - \Theta\phi_{_R} \end{pmatrix} = \mathcal{U}^- + i\mathcal{V}^- \quad . \tag{10b}$$

One can see that

$$\overset{MR}{v}(p^\mu) = \overset{MR}{\gamma}_5 \overset{MR}{u}(p^\mu) = i \overset{WR}{\gamma}_5 \overset{WR}{\gamma}_0 \overset{MR}{u}(p^\mu) = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix} \overset{MR}{u}(p^\mu) \quad . \tag{11}$$

The index *CR* stands for 'the canonical representation', *WR*, for 'the Weyl representation', *MR*, for 'the Majorana representation'. For the second-type spinors [6] $\lambda^{S,A}$ and $\rho^{S,A}$ in both the $j=1/2$ and $j=1$ case the use of the Majorana representation leads to the natural separation into the real and imaginary parts when referring to positive- (negative-) solutions.

The real and imaginary parts of the positive-energy $u-$ bispinors of the helicity ± 1 are the following (cf. with the $j=1/2$ case, see *Appendix*):

$$\mathcal{U}^+_\uparrow(p^\mu) = \mathcal{U}^+_\downarrow(p^\mu) = \frac{1}{2\sqrt{2}} \begin{pmatrix} p^- - \frac{p_2(p_1+p_2)}{E+m} \\ -\sqrt{2}(p_1 - \frac{p_2p_3}{E+m}) \\ p^+ + \frac{p_2(p_1-p_2)}{E+m} \\ p^- + \frac{p_2(p_1-p_2)}{E+m} \\ -\sqrt{2}(p_1 + \frac{p_2p_3}{E+m}) \\ p^+ - \frac{p_2(p_1+p_2)}{E+m} \end{pmatrix} \quad , \tag{12a}$$

$$\mathcal{V}^+_\uparrow(p^\mu) = -\mathcal{V}^+_\downarrow(p^\mu) = \frac{1}{2\sqrt{2}} \begin{pmatrix} -p^- + \frac{p_1(p_1+p_2)}{E+m} \\ -\sqrt{2}(p_2 + \frac{p_1p_3}{E+m}) \\ p^+ - \frac{p_1(p_1-p_2)}{E+m} \\ p^- - \frac{p_1(p_1-p_2)}{E+m} \\ -\sqrt{2}(p_2 - \frac{p_1p_3}{E+m}) \\ -p^+ + \frac{p_1(p_1+p_2)}{E+m} \end{pmatrix} \quad . \tag{12b}$$

Surprisingly, real (and imaginary) parts of bispinors of different helicities appear to be equal each other (within a sign). Thus, they are connected by the operation of the complex conjugation. As to the solution with $h=0$ one has only imaginary part of the positive-energy bispinor:

$$\mathcal{U}^+_\rightarrow(p^\mu) \equiv 0 \quad , \quad \mathcal{V}^+_\rightarrow(p^\mu) = \frac{1}{2} \begin{pmatrix} (p^- + m)\frac{p_1+p_2}{E+m} \\ -\sqrt{2}(m + \frac{p_1^2+p_2^2}{E+m}) \\ (p^+ + m)\frac{p_1-p_2}{E+m} \\ (p^- + m)\frac{p_2-p_1}{E+m} \\ \sqrt{2}(m + \frac{p_1^2+p_2^2}{E+m}) \\ -(p^+ + m)\frac{p_1+p_2}{E+m} \end{pmatrix} \quad . \tag{13}$$

The corresponding procedure can also be carried out for the negative-energy solutions; the "bispinors" are connected with (12a,12b,13) using the equation (11). Unlike to 'transversal' bispinors ($h=\pm 1$) the bispinor $v_\rightarrow(p^\mu)$ has only a real part.

Finally, one cannot find a matrix which transfers over the equation with pure imaginary matrices because the unit matrix commutes with all matrices of the unitary transformation.

In conclusion, using the standard form of the field operator in the x- representation one can separate out the real and imaginary parts of the $(1,0)\oplus(0,1)$ coordinate-space "bispinors", then the result can be compared with the case of the $(1/2,0)\oplus(0,1/2)$ representation. Relevant commutation relations can be found. However, if we would wish to obtain an entirely real coordinate-space equation (without any care of the $x-$ space imaginary part of the field function) such a procedure leads to certain constraints between

components of the 4-vector momentum and/or constraints on the phase factors. The physical interpretation of the latter statement is unobvious and should be searched in a separate paper.

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`Appendix.` Here we wish to present the explicit forms of $\mathcal{U}^{\pm}_{\uparrow\downarrow}$ and $\mathcal{V}^{\pm}_{\uparrow\downarrow}$, the real and imaginary parts of bispinors in the $j = 1/2$ Majorana representation. Comparing with the $j = 1$ case we can observe differences. The matrix of transfer over the Majorana representation from the Weyl representation is

$$U = \frac{1}{2}\begin{pmatrix} 1\!\!1 - i\Theta & 1\!\!1 + i\Theta \\ -1\!\!1 - i\Theta & 1\!\!1 - i\Theta \end{pmatrix} \quad , \quad U^{\dagger} = \frac{1}{2}\begin{pmatrix} 1\!\!1 - i\Theta & -1\!\!1 - i\Theta \\ 1\!\!1 + i\Theta & 1\!\!1 - i\Theta \end{pmatrix} \quad . \tag{14}$$

The γ- matrices are given by

$$\gamma^0_{_{MR}} = \begin{pmatrix} 0 & -i\Theta_{[1/2]} \\ -i\Theta_{[1/2]} & 0 \end{pmatrix} \quad , \quad \gamma^1_{_{MR}} = \begin{pmatrix} -i\sigma_1\Theta_{[1/2]} & 0 \\ 0 & -i\sigma_1\Theta_{[1/2]} \end{pmatrix} \tag{15a}$$

$$\gamma^2_{_{MR}} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad , \quad \gamma^3_{_{MR}} = \begin{pmatrix} -i\sigma_3\Theta_{[1/2]} & 0 \\ 0 & -i\sigma_3\Theta_{[1/2]} \end{pmatrix} \quad , \tag{15b}$$

$$\text{and} \quad \gamma^5_{_{MR}} = \begin{pmatrix} -i\Theta_{[1/2]} & 0 \\ 0 & i\Theta_{[1/2]} \end{pmatrix} \quad . \tag{15c}$$

All they are imaginary and are related with Eq. (2).

$$\mathcal{U}^{+}_{\uparrow}(p^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} E + m - p_2 \\ 0 \\ -p_3 \\ -p_1 \end{pmatrix} \quad , \quad \mathcal{U}^{+}_{\downarrow}(p^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} 0 \\ E + m - p_2 \\ -p_1 \\ p_3 \end{pmatrix} \; , \tag{16a}$$

$$\mathcal{V}^{+}_{\uparrow}(p^{\mu}) = \overset{WR}{\gamma_5}\overset{WR}{\gamma_0}\mathcal{U}^{+}_{\downarrow}(\tilde{p}^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} p_1 \\ -p_3 \\ 0 \\ -E - m - p_2 \end{pmatrix} \quad , \tag{16b}$$

$$\mathcal{V}^{+}_{\downarrow}(p^{\mu}) = -\overset{WR}{\gamma_5}\overset{WR}{\gamma_0}\mathcal{U}^{+}_{\uparrow}(\tilde{p}^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} -p_3 \\ -p_1 \\ E + m + p_2 \\ 0 \end{pmatrix} \quad . \tag{16c}$$

The negative-energy spinors are related with the positive-energy ones by using the formulas:

$$\overset{MR}{v_{\uparrow}}(p^{\mu}) = -i[\overset{MR}{u_{\downarrow}}(p^{\mu})]^{*} \quad , \quad \overset{MR}{v_{\downarrow}}(p^{\mu}) = +i[\overset{MR}{u_{\uparrow}}(p^{\mu})]^{*} \quad , \tag{17}$$

and, thus,

$$\mathcal{U}_{\uparrow}^{+} = \mathcal{V}_{\downarrow}^{-} = \Re e\, u_{\uparrow}^{^{MR}} = \frac{u_{\uparrow}^{^{MR}} - i v_{\downarrow}^{^{MR}}}{2} \quad , \quad \mathcal{U}_{\downarrow}^{+} = -\mathcal{V}_{\uparrow}^{-} = \Re e\, u_{\downarrow}^{^{MR}} = \frac{u_{\downarrow}^{^{MR}} + i v_{\uparrow}^{^{MR}}}{2} \quad , \tag{18a}$$

$$\mathcal{V}_{\uparrow}^{+} = \mathcal{U}_{\downarrow}^{-} = \Im m\, u_{\uparrow}^{^{MR}} = \frac{u_{\uparrow}^{^{MR}} + i v_{\downarrow}^{^{MR}}}{2i} \quad , \quad \mathcal{V}_{\downarrow}^{+} = -\mathcal{U}_{\uparrow}^{-} = \Im m\, u_{\downarrow}^{^{MR}} = \frac{u_{\downarrow}^{^{MR}} - i v_{\uparrow}^{^{MR}}}{2i} \quad . \tag{18b}$$

These formulas also can be used to form even- and odd- bispinors with respect to $\mathbf{p} \rightarrow -\mathbf{p}$.

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